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ABSTRACT

A study on sampling errors of variance components was conducted within the framework of generalizability theory by P. L. Smith (1978). The study used an intuitive approach for solving the problem of how to allocate the number of conditions to different facets in order to produce the most stable estimate of the universe score variance. Optimization techniques are proposed as a promising approach for solving allocation problems in generalizability studies. An important consequence of the conclusion that allocation problems in generalizability studies have much in common with allocation problems in decision studies would be a considerable reduction in the effort required to develop optimization procedures and computer programs. This examination of optimization techniques is the beginning of a procedure that enables the researcher to design a generalizability study in line with stated limits on sampling errors as well as resources. (SLD)

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SAMPLING ERRORS OF VARIANCE COMPONENTS

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Paper presented at the annual meeting of the National Council on Measurement in Education, Atlanta, April 1993

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Abstract

A major study on sampling errors of variance components was conducted within the framework of generalizability theory by Smith (1978). The study employed an intuitive approach for solving the problem of how to allocate the number of conditions to different facets in order to produce the most stable estimate of the universe score variance. In this study, optimization techniques are proposed as a promising approach for solving allocation problems in generalizability studies.

Key words: generalizability theory, variance components, mathematical programming, sampling errors.

Introduction

Generalizability theory (Cronbach, Gleser, Nanda, & Rajaratnam, 1972) makes a distinction between a generalizability study and a decision study. A generalizability study is conducted to obtain estimates of variance components associated with the universe of admissible observations. In a decision study these estimates are used to make decisions on the actual composition of a measurement instrument. Owing to limited resources, however, generalizability studies are often carried out with relatively small samples and relatively few conditions of each facet. Cronbach et al. (1972) have pointed out the danger of employing estimates of variance components and coefficients of generalizability studies based on small samples. The estimates of the variance components in these studies frequently entail sampling errors too large to be useful for a decision study.

A major study on sampling errors of variance components was conducted within the framework of generalizability theory by Smith (1978). The study employed an intuitive approach for solving the problem of how to allocate the number of conditions to different facets in order to produce the most stable estimate of the universe score variance. In recent studies, the versatility of optimization techniques for solving allocation problems in decision studies was amply demonstrated. In this paper, optimization techniques are proposed as a promising approach for solving allocation problems in generalizability studies.

Sampling errors of estimates of variance components are derived first. The minimization of sampling errors of variance components under constraints is discussed next. The balanced two-facet random-model crossed design will be used for illustration.

Sampling Errors of Estimates of Variance Components

In generalizability theory, the analysis of variance method (e.g., Searle, 1971, p. 384) is used for the estimation of variance components from balanced data. This method starts with calculating the mean squares of the analysis of variance, whereafter the expected values of the mean squares are derived under the random or mixed model. Table 1 contains the expected mean squares for the two-facet random-model crossed design.

TABLE 1

Expected Mean Squares for $p \times i \times j$ Random-Model Crossed Design

Source of Variance	Expected Mean Squares
Persons (p)	$EMS_p = \sigma^2_{pij,e} + n_i\sigma^2_{pj} + n_j\sigma^2_{pi} + n_in_j\sigma^2_p$
Items (i)	$EMS_i = \sigma^2_{pij,e} + n_p\sigma^2_{ij} + n_j\sigma^2_{pi} + n_pn_j\sigma^2_i$
Raters (j)	$EMS_j = \sigma^2_{pij,e} + n_p\sigma^2_{ij} + n_i\sigma^2_{pj} + n_pn_i\sigma^2_j$
Persons x items (pi)	$EMS_{pi} = \sigma^2_{pij,e} + n_j\sigma^2_{pi}$
Persons x raters (pj)	$EMS_{pj} = \sigma^2_{pij,e} + n_i\sigma^2_{pj}$
Items x raters (ij)	$EMS_{ij} = \sigma^2_{pij,e} + n_p\sigma^2_{ij}$
Residual (pij,e)	$EMS_{res} = \sigma^2_{pij,e}$

Equating the expected values to the calculated values leads to linear equations in the variance components, the solutions to which are the estimators of the variance components. The variance components for the two-facet random-model crossed design can be obtained with the following equations:

$$\hat{\sigma}^2_{pij,e} = MS_{pij,e}$$
$$\hat{\sigma}^2_{ij} = (MS_{ij} - MS_{pij,e})/n_p$$
$$\hat{\sigma}^2_{pj} = (MS_{pj} - MS_{pij,e})/n_i$$
$$\hat{\sigma}^2_{pi} = (MS_{pi} - MS_{pij,e})/n_j$$
$$\hat{\sigma}^2_{j} = (MS_{j} - MS_{ij} - MS_{pj} + MS_{pij,e})/(n_pn_i)$$
$$\hat{\sigma}^2_{i} = (MS_{i} - MS_{ij} - MS_{pi} + MS_{pij,e})/(n_pn_j)$$
$$\hat{\sigma}^2_{p} = (MS_{p} - MS_{pj} - MS_{pi} + MS_{pij,e})/(n_in_j)$$

In the statistical literature (e.g., Searle, 1971, pp. 415-417), the theoretical sampling variance of estimates of variance components is expressed in terms of the expected mean squares. As was shown in Table 1, the expected mean squares are linear combinations of the population values of the variance components, and therefore the expected sampling variance of the variance components of the two-facet random-model crossed design can be expressed as:

$$var\ \hat{\sigma}_p^2 = \frac{2}{(n_p-1)}\left[\left(\sigma_p^2 + \frac{\sigma_{pj}^2}{n_j} + \frac{\sigma_{pi}^2}{n_i} + \frac{\sigma_{res}^2}{n_i n_j}\right)^2 + \frac{1}{(n_i-1)}\left(\frac{\sigma_{pi}^2}{n_i} + \frac{\sigma_{res}^2}{n_i n_j}\right)^2 \right. \tag{1a}$$

$$\left. + \frac{1}{(n_j-1)}\left(\frac{\sigma_{pj}^2}{n_j} + \frac{\sigma_{res}^2}{n_i n_j}\right)^2 + \frac{1}{(n_i-1)(n_j-1)}\left(\frac{\sigma_{res}^2}{n_i n_j}\right)^2\right],$$

$$var\ \hat{\sigma}_i^2 = \frac{2}{(n_i-1)}\left[\left(\sigma_i^2 + \frac{\sigma_{pi}^2}{n_p} + \frac{\sigma_{ij}^2}{n_j} + \frac{\sigma_{res}^2}{n_p n_j}\right)^2 + \frac{1}{(n_p-1)}\left(\frac{\sigma_{pi}^2}{n_p} + \frac{\sigma_{res}^2}{n_p n_j}\right)^2 \right. \tag{1b}$$

$$\left. + \frac{1}{(n_j-1)}\left(\frac{\sigma_{ij}^2}{n_j} + \frac{\sigma_{res}^2}{n_p n_j}\right)^2 + \frac{1}{(n_p-1)(n_j-1)}\left(\frac{\sigma_{res}^2}{n_p n_j}\right)^2\right],$$

$$var\ \hat{\sigma}_j^2 = \frac{2}{(n_j-1)}\left[\left(\sigma_j^2 + \frac{\sigma_{pj}^2}{n_p} + \frac{\sigma_{ij}^2}{n_i} + \frac{\sigma_{res}^2}{n_p n_i}\right)^2 + \frac{1}{(n_p-1)}\left(\frac{\sigma_{pj}^2}{n_p} + \frac{\sigma_{res}^2}{n_p n_i}\right)^2 \right. \tag{1c}$$

$$\left. + \frac{1}{(n_i-1)}\left(\frac{\sigma_{ij}^2}{n_i} + \frac{\sigma_{res}^2}{n_p n_i}\right)^2 + \frac{1}{(n_p-1)(n_i-1)}\left(\frac{\sigma_{res}^2}{n_p n_i}\right)^2\right],$$

$$var\ \hat{\sigma}_{pj}^2 = \frac{2}{(n_p-1)(n_j-1)}\left[\left(\sigma_{pj}^2 + \frac{\sigma_{res}^2}{n_i}\right)^2 + \frac{1}{(n_i-1)}\left(\frac{\sigma_{res}^2}{n_i}\right)^2\right], \tag{1d}$$

$$var\ \hat{\sigma}_{pi}^2 = \frac{2}{(n_p-1)(n_i-1)}\left[\left(\sigma_{pi}^2 + \frac{\sigma_{res}^2}{n_j}\right)^2 + \frac{1}{(n_j-1)}\left(\frac{\sigma_{res}^2}{n_j}\right)^2\right], \tag{1e}$$

$$var\ \hat{\sigma}_{ij}^2 = \frac{2}{(n_i-1)(n_j-1)}\left[\left(\sigma_{ij}^2 + \frac{\sigma_{res}^2}{n_p}\right)^2 + \frac{1}{(n_p-1)}\left(\frac{\sigma_{res}^2}{n_p}\right)^2\right], \tag{1f}$$

$$var\ \hat{\sigma}_{res}^2 = \frac{2}{(n_p-1)(n_j-1)(n_i-1)}\sigma_{res}^4. \tag{1g}$$

The three factors affecting the magnitude of the sampling errors of the variance components can be inferred from equations (1a-1g).

The first factor concerns the relative magnitudes of the population values of variance components. While the population values of the variance components are not known, equations (1a-1g) can be very useful for a researcher planning a generalizability study if prior information on the relative magnitude of the population values of variance components is available from comparable studies. In Table 2, for example, total variance is proportioned

in a way that resembles estimates commonly observed in applications of the two-facet design with respect to judgmental data such as observations or ratings (Smith, 1976, p. 90). The residual component is represented by a small ($\sigma^2_{res} = 20$) and a large ($\sigma^2_{res} = 76$) error component.

TABLE 2

Variance Components of a Two-Facet Random-Model Crossed Design

Source of Variance	Variance Components	Proportion of total variance small error	large error
Persons (*p*)	$\sigma^2_p = 20$	.09	.07
Items (*i*)	$\sigma^2_i = 47$	.21	.17
Raters (*j*)	$\sigma^2_j = 12$	.05	.04
Persons x items (*pi*)	$\sigma^2_{pi} = 64$	.29	.23
Persons x raters (*pj*)	$\sigma^2_{pj} = 35$	.16	.13
Items x raters (*ij*)	$\sigma^2_{ij} = 23$	.10	.08
Residual (*pij,e*)	$\sigma^2_{res} = 20{,}76$	.09	.27

The effect of the second factor, design configuration, on the magnitude of the sampling error of the universe score variance can be seen by comparing equation (1a) to equation (2) for a design which has facet J nested within facet I:

$$var\ \hat{\sigma}^2_p = \frac{2}{(n_p-1)}\left[\left(\sigma^2_p + \frac{\sigma^2_{pi}}{n_i} + \frac{\sigma^2_{res}}{n_j n_i}\right)^2 + \frac{1}{(n_i-1)}\left(\frac{\sigma^2_{pi}}{n_i} + \frac{\sigma^2_{res}}{n_j n_i}\right)^2\right]. \qquad (2)$$

The comparison of the two equations indicates that the sampling error of the estimate of the universe score variance can be decreased by employing a nested instead of a crossed design. The third factor concerns the number of measurement objects and the number of conditions of each facet. It is obvious that a researcher can avoid large sampling errors by employing large samples and large numbers of observations in a generalizability study. According to Smith (1978, p. 322), however, many studies use rather small sample sizes. In the designs examined in his study, the number of subjects (n_p) were represented at three magnitudes: low (n_p

= 25), medium (n_p = 50), and high (n_p = 100). The number of conditions of each of the two facets (n_i and n_j) were represented at two, four, and eight conditions.

Minimizing Sampling Errors of Estimates of Variance Components

Three factors affecting the sampling errors of variance components were identified above. However, a researcher cannot influence the first factor, and the second factor implies changing the design of the generalizability study. Therefore, given a certain design choice, a researcher can only manipulate the third factor. In the following, two approaches for controlling these sampling errors are presented. The results of the approach followed by Smith (1976, 1978) will be examined first. An approach for solving allocation problems based on mathematical programming is presented next.

Intuitive Approach

Under the constraints of a fixed number of n_p subjects, and a fixed number of $L = n_i n_j$ observations, Smith (1978) sought the solution for the problem of how to allocate the observations in order to minimize the sampling variance of the universe score variance. Since, according to Smith (1978, p.328), simple solutions for this minimization problem could not be derived analytically, a more intuitive approach was chosen. From the examination of equation (1a) he concluded that as the residual component increases, equal allocation will maximize the divisor of the residual component and the optimal solution tends toward $n_i = n_j$. From a study by Woodward and Joe (1973) he borrowed the conclusion that as the residual component decreases, the allocation to n_i and n_j is determined by the ratio of σ^2_{pi} and σ^2_{pj}. In order to evaluate these two conclusions, the minimization problem is stated in terms of mathematical programming as

minimize $var\ \hat{\sigma}^2_p$ objective-function (3)

subject to $L = n_i n_j$, and

n_p = constant. equality constraints (4)

Objective-function (3) can be composed of four parts:

$$\left(\sigma_p^2 + \frac{\sigma_{pi}^2}{n_i} + \frac{\sigma_{pj}^2}{n_j} + \frac{\sigma_{res}^2}{n_i n_j}\right)^2, \tag{3a}$$

$$\frac{1}{(n_i-1)}\left(\frac{\sigma_{pi}^2}{n_i} + \frac{\sigma_{res}^2}{n_i n_j}\right)^2, \tag{3b}$$

$$\frac{1}{(n_j-1)}\left(\frac{\sigma_{pj}^2}{n_j} + \frac{\sigma_{res}^2}{n_i n_j}\right)^2, \tag{3c}$$

$$\frac{1}{(n_i-1)(n_j-1)}\left(\frac{\sigma_{res}^2}{n_i n_j}\right)^2. \tag{3d}$$

From equations (3a - 3d) it can be inferred that (3a), the 'persons' part, determines the value of the objective-function almost completely. With the appropriate variance components from Table 2 with a small error component and n_i = 8 and n_j = 4, the contribution of (3a) to the value of the objective-function is 97%. With a large error component, this contribution is 96%. For both error components, the contribution of (3d), the 'residual' part, is negligible. The optimal continuous solutions for the variance components from Table 2 with L = 32 and σ_{res}^2 equal to 20 or 76, are computed as n_i = 7.4 and n_j = 4.3. The same solutions are obtained with σ_{res}^2 equal to zero. With an improbable residual component equal to 200, however, the optimal continuous solutions are only slightly different, n_i = 7.3 and n_j = 4.4. Only an infinitely large residual component would affect the value of the objective-function and result in a solution tending toward $n_i = n_j$. The same optimal integer solutions, n_i = 8 and n_j = 4, are obtained with different residual components. The conclusion therefore is that the optimal number of conditions for the two facets does not depend on the magnitude of the residual component. To a lesser degree the same conclusion applies to parts (3b) and (3c), the 'interaction' components, of equation (1a).

Woodward and Joe (1973) showed that for the problem of maximizing the generalizability coefficient subject to the constraint that the total number of observations is fixed, the optimal allocation of conditions should be made directly proportional to $\sigma_{pi}^2/\sigma_{pj}^2$. In Sanders, Theunissen, and Baas (1991), their problem was stated in terms of mathematical programming as

$$\text{minimize} \quad \frac{\sigma^2_{pi}}{n_i} + \frac{\sigma^2_{pj}}{n_j} + \frac{\sigma^2_{res}}{n_i n_j} \qquad \text{objective-function} \qquad (5)$$

$$\text{subject to } L = n_i n_j . \qquad \text{equality constraint} \qquad (6)$$

Because of equality constraint (6), the residual component does not play a role in determining the optimal number of conditions. The optimal continuous solutions can be shown to be

$$n_i = \left(L \frac{\sigma^2_{pi}}{\sigma^2_{pj}} \right)^{1/2}, \text{ and } n_j = \left(L \frac{\sigma^2_{pj}}{\sigma^2_{pi}} \right)^{1/2} .$$

Smith (1978, p. 329) fails to explain why the results obtained for the optimization problem defined by (5) and (6) also apply to a seemingly quite different optimization problem as defined by (3) and (4). The reason is that the dominant role of part (3a) in objective-function (3) makes the problem of minimizing objective-function (3) almost identical to the problem of minimizing objective-function (5). Note that by taking the square root of (3a) and leaving out the constant σ^2_p, equation (5) is obtained. The continuous solutions for the minimization problem defined by (3) and (4) will therefore hardly differ from those for the minimization problem defined by (5) and (6). The optimal continuous solutions for the optimization problem defined by (5) and (6) for the variance components from Table 2 are $n_i = 7.6$ and $n_j = 4.2$.

Mathematical Programming Approach

Mathematical programming can be used to solve the problem of controlling the sampling errors of variance components. The allocation problem is viewed as the minimization of sampling errors under constraints regarding observations and as the minimization of observations under constraints regarding sampling errors.

From the foregoing discussion the conclusion can be drawn that for most generalizability studies the continuous solutions of optimization problems defined by (3) and (4) will not be much different from the continuous solutions of optimization problems defined by (5) and (6). A procedure for solving the optimization problem defined by (5) and (6) was presented in Sanders, Theunissen, and Baas (1991). This procedure, which also allows for less restrictive and more realistic constraints than (6), can be applied to the optimization problem under consideration.

Smith (1978, p. 327) only discusses the problem of minimizing the sampling error of the universe score variance. Mathematical programming, however, offers the possibility to solve optimization problems with multiple objective-functions. Williams (1985, pp. 24-26) describes three ways of dealing with multiple objective-functions. First, the optimization problem can be solved with each objective-function in turn. A satisfactory solution for the optimization problem can be selected from the solutions resulting from these different objective-functions. Second, objectives and constraints can be interchanged, that is, treating all but one objective-function as a constraint. Third, a linear combination of all the objective-functions and optimize the multiple objective-function can be taken. The optimization problem

$$\text{minimize} \quad w_p var\ \hat{\sigma}_p^2 + w_i var\ \hat{\sigma}_i^2 + w_j var\ \hat{\sigma}_j^2 \qquad \text{objective-function}$$

$$\text{subject to } L = n_i n_j, \text{ and}$$
$$n_p = \text{constant}, \qquad \text{equality constraints}$$

is a version of the optimization problem defined by (3) and (4) with (3) now being a linear combination of three objective-functions. This solution to the optimization problem requires giving relative weights, w_p, w_i, and w_j, to the three objective-functions. A less restrictive formulation of this multi-objective optimization problem is obtained by introducing an inequality constraint for the sample size, $n_p \leq 100$. It can be shown that inequality constraints have many advantages over equality constraints. According to Williams (1985, p. 26), there is no one obvious way of dealing with multiple objectives. Some or all of the three methods should be used for certain optimization problems.

The other approach for controlling sampling errors of variance components consists of minimizing the number of observations and/or the number of measurement objects under constraints of specific sampling errors. A general formulation of this optimization problem with one objective-function is

$$\text{minimize} \quad n_p n_i n_j \qquad \text{objective-function}$$

$$\text{subject to } var\ \hat{\sigma}_p^2 \leq e_p, \qquad \text{threshold constraint}$$

where e_p stands for the value of the sampling error of the universe score variance considered to be acceptable. In order to gain an idea of the magnitudes of the sampling errors of variance components, Smith (1978, p. 335) suggested comparing the sampling errors obtained by equation (1a) with those that would be expected for variance estimates obtained by drawing repeated samples from a single normal distribution. Assuming a fixed number of subjects, a restricted version of the foregoing optimization problem is

minimize $n_i n_j$ objective-function

subject to $var\ \hat{\sigma}_p^2 \leq e_p$. threshold constraint

Assuming that the sampling error of the universe score variance is defined by equation (5), solutions for this optimization problem can be obtained by the optimization procedure presented in Sanders, Theunissen, and Baas (1989). That procedure could also be used to obtain starting solutions for optimization problems with more complex objective-functions and/or constraints.

An example of an optimization problem with multiple constraints is

minimize $n_p n_i n_j$ objective-function

subject to $var\ \hat{\sigma}_p^2 \leq e_p$, and
$var\ \hat{\sigma}_i^2 \leq e_i$, and
$var\ \hat{\sigma}_j^2 \leq e_j$, threshold constraints

where e_p, e_i, and w_j, stand for the values of three sampling errors of variance components. In general, the value for e_p will be smaller than the value for e_i or e_j. Because the size of the sample is often the factor responsible for the major part of the costs of a generalizability study, the optimization problem could be formulated as

minimize $n_i n_j$ objective-function

subject to $n_p \leq L$ threshold constraints

$var\ \hat{\sigma}_p^2 \leq e_p$, and

$var\ \hat{\sigma}_i^2 \leq e_i$, and

$var\ \hat{\sigma}_j^2 \leq e_j$,

where L is the maximum sample size that the researcher can afford.

Conclusions and Discussion

In the study presented in this paper it was shown how our understanding of factors determining the sampling errors of variance components can be improved by viewing the problem of the optimal allocation of observations as an optimization problem. This approach leads to the important conclusion that allocation problems in generalizability studies have much in common with allocation problems in decision studies. An important practical consequence of this conclusion could be a considerable reduction of the effort required to develop optimization procedures and computer programs.

Apart from the three factors discussed before, another factor affecting the magnitude of the sampling errors of variance components is the complexity of the design (Smith, 1981). The effect of this factor on the stability of the estimates can be shown by comparing equation (1a) to the equation for a one-facet random-model crossed (p x i) design:

$$var\ \hat{\sigma}_p^2 = \frac{2}{(n_p-1)}\left[\left(\sigma_p^2 + \frac{\sigma_{res}^2}{n_i}\right)^2 + \frac{1}{(n_i-1)}\left(\frac{\sigma_{res}^2}{n_i}\right)^2\right].$$

From the comparison it is clear that the sampling error of the universe score variance component of the one-facet random-model will be smaller than the sampling error of the universe score variance of the two-facet random-model. However, there is an essential difference between the impact of this factor and the other three factors. Design complexity implies generalizations to different universes which means that the variance component estimates have different interpretations.

The results of Smith's study (1978, p. 336) show that the sampling errors for the universe score variance are intolerably large for practical purposes unless $n_p n_i n_j$ is 800 or

larger. Bell (1986) found the same results in a study on simultaneous confidence intervals of variance components. It should be clear to the researcher planning a generalizability study that the problem of sampling errors may make the results of the study unusable. Another problem is that of limited resources (time, money, etc.) which may render the study infeasible. What is needed therefore is a procedure which enables the researcher to design a generalizability study in line with stated limits on sampling errors as well as resources. The start of such a procedure was presented in this paper.

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